Notes Feb 12.

① Finite type in regular coord's.

Let $M \subseteq \mathbb{C}^{n+1}$ be real hypersurface, $p \in M$. Choose regular coord's $(z,w) \in \mathbb{C}^n \times \mathbb{C}$ s.t. $p = (0,0)$ and M is given by

$$\operatorname{Im} w = \varphi(z, \bar{z}, \operatorname{Re} w).$$

Recall that a frame for CR v.f. is

$$\bar{L}_j = \frac{\partial}{\partial \bar{z}_j} - 2i \frac{\varphi_{\bar{z}_j}}{1 + i\varphi_s} \frac{\partial}{\partial \bar{w}}, \quad j = 1, \dots, n$$

Prop 1. TFAE

(i) $\varphi(z, \bar{z}, s) = P(z, \bar{z}) + O(|z|^{m+1}, s)$, where P is a homogeneous polynomial of $\deg = m$ and $\partial\bar{\partial} P \not\equiv 0$.

(ii) M is of finite type at p w/ H-type $= m$.

Sketch of pf. We note that | For simplicity, we assume that M is real analytic.

$$\frac{\varphi_{z_i}}{1+i\varphi_s} = P_{z_i} + O(|z|^m, s)$$

No linear term in s since $d\varphi(0)=0$.

We compute:

$$[Z_i, \overline{Z}_j] = -2i\, P_{z_i \bar{z}_j} \frac{\partial}{\partial w} - 2i\, P_{z_i \bar{z}_j} \frac{\partial}{\partial \bar{w}} + O(|z|^{m-1}, s)$$

homog. in $z, \bar{z}$ of deg $= m-2$.

Point is when we apply $\frac{\partial}{\partial w}$ or $\frac{\partial}{\partial \bar{w}}$, we also multiply by T, and thus the only term getting "lower degree" is the $\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}$ derivatives on P.

We also note that (by formal integrability) $[Z_i, Z_j]$ and $[\overline{Z_i}, \overline{Z_j}]$ are both 0 in this case. Thus, any repeated commutator evaluated at 0, must have at least one $\frac{\partial}{\partial z_i}$ and one $\frac{\partial}{\partial \overline{z_j}}$ on P unless it vanishes at $p = (0,0)$. If we write

$$P(z,\overline{z}) = \underbrace{Q(z)}_{\text{holom. poly. in } z.} + \overline{Q(z)} + \underbrace{R(z,\overline{z})}_{\text{Poly. } \sum_{\substack{|\alpha|+|\beta|=m \\ \alpha\neq 0, \beta\neq 0}} c_{\alpha\beta} z^\alpha \overline{z^\beta}}$$

By the obs. above, in any relevant repeated commutator the terms $Q(z) + \overline{Q(z)}$ will vanish. By repeating the commutator calculation on previous page $\longrightarrow$

we see that if $R(z,\bar{z})$ has a nontrivial term $z^\alpha \bar{z}^\beta$, $\alpha \neq 0, \beta \neq 0$. $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$ then the commutator

$$\left[\cdots \left[\left[Z_{\alpha_1}, \bar{Z}_{\beta_1} \right], Z_{\alpha_2} \right] \cdots \right] \Big|_0 =$$

$$a \frac{\partial}{\partial w} + b \frac{\partial}{\partial \bar{w}}, \quad \text{with } ab \neq 0.$$

This proves (a) $\Rightarrow$ (b).

For (b) $\Rightarrow$ (a), we assume that (a) does not hold. Then (since φ real anal.)

$$\varphi(z, \bar{z}, s) = f(z) + \overline{f(z)} + O(s),$$

where f is a holomorphic fcn w/o linear terms.

meaning each term in Taylor has an s factor.

Consider the bihol. change of variables

$$z = z', \quad w = w' + 2i f(z')$$

$$\operatorname{Im} w = \frac{1}{2i}(w - \bar{w}) = \frac{1}{2i}(w' - \bar{w}') +$$
$$f(z') + \overline{f(z')}.$$

Plugging this into the defining eq'n

$$\operatorname{Im} w = f(z) + \overline{f(z)} + O(s)$$

We get

$$\operatorname{Im} w' = O(s).$$

Now, the calculations above yield that any commutator, evaluated at 0, is in fact 0. Thus, M is not of finite type. $\boxtimes$

② Nonminimal h-surf. in $\mathbb{C}^{n+1}$.

Recall. A CR mfld $(M^{2n+d}, \mathcal{V})$ is nonminimal at p if $\exists$ CR mfld M' w/ $\dim M' < \dim M$ while $\mathcal{V} \subseteq \mathbb{C} \otimes TM'$. We have

M finite type at $p \Rightarrow M$ minimal at p. Converse is not true for smooth CR mflds, but is true for real-analytic ones.

Prop 2. Let $M \subseteq \mathbb{C}^{n+1}$ be a real hypersurface (CR codim $= d = 1$) and $p \in M$. If M is nonminimal at p, then $\exists$ cplx hypersurface (cplx submfld of dim n) $E \subseteq M$ w/ $p \in E$.

Sketch of pf Let $E \subseteq M$, $p \in E$, loc CR mfld s.t. $T^{1,0}M, T^{0,1}M \subseteq \mathbb{C} \otimes TE$, and $\dim E < \dim M$. Well, the only possibility then is that $\dim E = 2n$ and then $\mathbb{C} \otimes TE = T^{1,0}_M + T^{0,1}_M$.

The conclusion follows from the following Prop from [BER]:

Prop 3. Let $M \subseteq \mathbb{C}^N$ be CR submfld. TFAE

(i) M is cplx (holom.) submfld.

(ii) $H = TM$

(iii) $T^{1,0}M + T^{0,1}M = \mathbb{C} \otimes TM$

(iv) $\dim M = \mathrm{CR}\dim M$

Pf: (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) is pretty clear. One completes by showing, e.g. (ii) $\Rightarrow$ (i). For CR submflds in $\mathbb{C}^N$ this is elementary (essentially linear algebra); see Prop 1.3.14 in [BER]. For abstract CR mflds, this is the Newlander-Nirenberg Thm.

$\square$